Langlands reciprocity for the even dimensional noncommutative tori

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Abstract

We conjecture an explicit formula for the higher dimensional Dirichlet character; the formula is based on the K-theory of the so-called noncommutative tori. It is proved, that our conjecture is true for the two-dimensional and one-dimensional (degenerate) noncommutative tori; in the second case, one gets a noncommutative analog of the Artin reciprocity law.

Key words and phrases: Langlands program, noncommutative tori

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Introduction

The aim of the underlying note is to bring some evidence in favor of the following analog of the Langlands reciprocity [5]:

Conjecture 1 (Langlands conjecture for noncommutative tori) *Let K be a finite extension of the rational numbers $\mathbb{Q}$ with the Galois group $Gal\ (K|\mathbb{Q})$; for an irreducible representation $\sigma_{n+1} : Gal\ (K|\mathbb{Q}) \to GL_{n+1}(\mathbb{C})$, there exists a $2n$-dimensional noncommutative torus with real multiplication, $\mathcal{A}^{2n}_{RM}$, such that $L(\sigma_{n+1}, s) \equiv L(\mathcal{A}^{2n}_{RM}, s)$, where $L(\sigma_{n+1}, s)$ is the Artin L-function and $L(\mathcal{A}^{2n}_{RM}, s)$ an L-function attached to the $\mathcal{A}^{2n}_{RM}$. Moreover, $\mathcal{A}^{2n}_{RM}$ is the image of an n-dimensional abelian variety $V_n(K)$ under the (generalized) Teichmüller functor F_n.*

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For the notation and terminology we refer the reader to sections 1 and 3; the noncommutative torus $\mathcal{A}_{RM}^{2n}$ can be regarded as a substitute of the "automorphic cuspidal representation $\pi_{\sigma_{n+1}}$ of the group $GL(n+1)$" in terms of the Langlands theory. Roughly speaking, conjecture 1 says, that the Galois extensions of the field of rational numbers come from the even dimensional noncommutative tori with real multiplication. Note, that the noncommutative tori are intrinsic to the problem, since they classify the irreducible (infinite-dimensional) representations of the Lie group $GL(n+1)$ [11]; such representations are the heart of the Langlands program [5]. Our conjecture is supported by the following evidence.

Theorem 1 *Conjecture 1 is true for $n = 1$ (resp., $n = 0$) and K abelian extension of an imaginary quadratic field k (resp., the rational field $\mathbb{Q}$).*

The structure of the note is as follows. A minimal necessary notation is introduced in section 1 and a brief summary of the Teichmüller functor(s) is given in section 3. Theorem 1 is proved in section 2.

1 Preliminaries

1.1 Noncommutative tori

A. The k-dimensional noncommutative tori ([4],[12]). A *noncommutative k-torus* is the universal C^*-algebra generated by k unitary operators $u_1, \dots, u_k$; the operators do not commute with each other, but their commutators $u_i u_j u_i^{-1} u_j^{-1}$ are fixed scalar multiples $\exp\ (2\pi i \theta_{ij}),\ \theta_{ij} \in \mathbb{R}$ of the identity operator. The k-dimensional noncommutative torus, $\mathcal{A}_\Theta^k$, is defined by a skew symmetric real matrix $\Theta = (\theta_{ij}),\ 1 \le i, j \le k$. Further, we think of the $\mathcal{A}_\Theta^k$ as a noncommutative topological space, whose algebraic K-theory yields $K_0(\mathcal{A}_\Theta^k) \cong \mathbb{Z}^{2^{k-1}}$ and $K_1(\mathcal{A}_\Theta^k) \cong \mathbb{Z}^{2^{k-1}}$. The canonical trace τ on the C^*-algebra $\mathcal{A}_\Theta^k$ defines a homomorphism from $K_0(\mathcal{A}_\Theta^k)$ to the real line $\mathbb{R}$; under the homomorphism, the image of $K_0(\mathcal{A}_\Theta^k)$ is a $\mathbb{Z}$-module, whose generators $\tau = (\tau_i)$ are polynomials in θ_{ij}. (More precisely, $\tau = \exp(\Theta)$, where the exterior algebra of θ_{ij} is nilpotent.) Recall, that the C^*-algebras $\mathcal{A}$ and $\mathcal{A}'$ are said to be stably isomorphic (Morita equivalent), if $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{A}' \otimes \mathcal{K}$ for the C^*-algebra $\mathcal{K}$ of compact operators; such an isomorphism indicates, that the C^*-algebras are homeomorphic as noncommutative topological spaces. By a result of Rieffel and Schwarz [13], the noncommutative tori $\mathcal{A}_\Theta^k$ and

$\mathcal{A}^k_{\Theta'}$ are stably isomorphic, if the matrices Θ and Θ' belong to the same orbit of a subgroup $SO(k,k \mid \mathbb{Z})$ of the group $GL_{2k}(\mathbb{Z})$, which acts on Θ by the formula $\Theta' = (A\Theta + B) \ / \ (C\Theta + D)$, where $(A,B,C,D) \in GL_{2k}(\mathbb{Z})$ and the matrices $A,B,C,D \in GL_k(\mathbb{Z})$ satisfy the conditions:

$$A^tD + C^tB = I, \quad A^tC + C^tA = 0 = B^tD + D^tB. \tag{1}$$

(Here I is the unit matrix and t at the upper right of a matrix means a transpose of the matrix.) The group $SO(k,k \mid \mathbb{Z})$ can be equivalently defined as a subgroup of the group $SO(k,k \mid \mathbb{R})$ consisting of linear transformations of the space $\mathbb{R}^{2k}$, which preserve the quadratic form $x_1x_{k+1}+x_2x_{k+2}+\ldots+x_kx_{2k}$.

B. The even dimensional normal tori. Further, we restrict to the case $k = 2n$ (the even dimensional noncommutative tori). It is known, that by the orthogonal linear transformations every (generic) real even dimensional skew symmetric matrix can be brought to the normal form:

$$\Theta_0 = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix} \tag{2}$$

where $\theta_i > 0$ are linearly independent over $\mathbb{Q}$. We shall consider the noncommutative tori $\mathcal{A}^{2n}_{\Theta_0}$, given by the matrix (2); we refer to the family as a *normal family*. Recall, that any noncommutative torus has a canonical trace τ, which defines a homomorphism from $K_0(\mathcal{A}^k_\Theta) \cong \mathbb{Z}^{2^{k-1}}$ to $\mathbb{R}$; it follows from [4], that the image of $K_0(\mathcal{A}^{2n}_{\Theta_0})$ under the homomorphism has a basis, given by the formula $\tau(K_0(\mathcal{A}^{2n}_{\Theta_0})) = \mathbb{Z} + \theta_1\mathbb{Z} + \ldots + \theta_n\mathbb{Z} + \sum_{i=n+1}^{2^{2n-1}} p_i(\theta)\mathbb{Z}$, where $p_i(\theta) \in \mathbb{Z}[1,\theta_1,\ldots,\theta_n]$.

C. The real multiplication ([8]). The noncommutative torus $\mathcal{A}^k_\Theta$ is said to have a *real multiplication*, if the endomorphism ring $End\ (\tau(K_0(\mathcal{A}^k_\Theta)))$ exceeds the trivial ring $\mathbb{Z}$. Since any endomorphism of the $\mathbb{Z}$-module $\tau(K_0(\mathcal{A}^k_\Theta))$ is the multiplication by a real number, it is easy to deduce, that all the entries of $\Theta = (\theta_{ij})$ are algebraic integers. (Indeed, the endomorphism is described by an integer matrix, which defines a polynomial equation involving θ_{ij}.) Thus, the noncommutative tori with real multiplication is a countable subset of all k-dimensional tori; any element of the set we shall denote by $\mathcal{A}^k_{RM}$. Notice, that for the even dimensional normal tori with real multiplication,

the polynomials $p_i(\theta)$ produce the algebraic integers in the extension of $\mathbb{Q}$ by θ_i; any such an integer is a linear combination (over $\mathbb{Z}$) of the θ_i. Thus, the trace formula reduces to $\tau(K_0(\mathcal{A}^{2n}_{RM})) = \mathbb{Z} + \theta_1\mathbb{Z} + \ldots + \theta_n\mathbb{Z}$.

1.2 L-function of noncommutative tori

We consider even dimensional normal tori with real multiplication. Denote by A a positive integer matrix, whose (normalized) Perron-Frobenius eigenvector coincides with the vector $\theta = (1, \theta_1, \ldots, \theta_n)$ and A is not a power of a positive integer matrix; in other words, $A\theta = \lambda_A\theta$, where $A \in GL_{n+1}(\mathbb{Z})$ and λ_A is the corresponding eigenvalue. (Explicitly, A can be obtained from vector θ as the matrix of minimal period of the Jacobi-Perron continued fraction of θ [2].) Let p be a prime number; take the matrix A^p and consider its characteristic polynomial $char\ (A^p) = x^{n+1} + a_1x^n + \ldots + a_nx + 1$. We introduce the following notation:

$$L_p^{n+1} := \begin{pmatrix} a_1 & a_2 & \ldots & a_n & p \\ -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 0 \end{pmatrix}. \tag{3}$$

A *local* zeta function of the $\mathcal{A}^{2n}_{RM}$ is defined as the reciprocal of $det\ (I_{n+1} - L_p^{n+1}z)$; in other words,

$$\zeta_p(\mathcal{A}^{2n}_{RM}, z) := \frac{1}{1 - a_1z + a_2z^2 - \ldots - a_nz^n + pz^{n+1}}, \quad z \in \mathbb{C}. \tag{4}$$

An *L-function* of $\mathcal{A}^{2n}_{RM}$ is a product of the local zetas over all, but a finite number, of primes $L(\mathcal{A}^{2n}_{RM}, s) = \prod_{p\ \nmid\ tr^2(A)-(n+1)^2} \zeta_p(\mathcal{A}^{2n}_{RM}, p^{-s}),\ s \in \mathbb{C}$.

Remark 1 *It will be shown, that for $n = 0$ and $n = 1$ formula (4) fits conjecture 1; for $n \geq 2$ it is an open problem based on an observation, that the crossed product $\mathcal{A}^{2n}_{RM} \rtimes_{L_p^{n+1}} \mathbb{Z}$ is a proper noncommutative analog of the (higher dimensional) Tate module, where matrix L_p^{n+1} corresponds to the Frobenius automorphism of the module [14], p.172.*

2 Proof of theorem 1

2.1 Case $n = 1$

Each one-dimensional abelian variety is a non-singular elliptic curve; choose this curve to have complex multiplication by (an order in) the imaginary quadratic field k and denote such a curve by E_{CM}. Then, by theory of complex multiplication, the (maximal) abelian extension of k coincides with the minimal field of definition of the curve E_{CM}, i.e. $E_{CM} \cong E(K)$ [14]. The Teichmüller functor $F := F_1$ maps $E(K)$ into a two-dimensional noncommutative torus with real multiplication (section 3); we shall denote the torus by $\mathcal{A}^2_{RM}$. To calculate the corresponding L-function $L(\mathcal{A}^2_{RM}, s)$, let A be a 2×2 positive integer matrix, whose normalized Perron-Frobenius eigenvenctor is $(1, \theta_1)$. For a prime p, the characteristic polynomial of the matrix A^p writes as $char\ (A^p) = x^2 + tr\ (A^p)x + 1$ and the matrix L^2_p takes the form:

$$L^2_p = \begin{pmatrix} tr\ (A^p) & p \\ -1 & 0 \end{pmatrix}. \tag{5}$$

The corresponding local zeta function $\zeta_p(\mathcal{A}^2_{RM}, z) = (1 - tr\ (A^p)z + pz^2)^{-1}$. We have to prove, that $\zeta_p(\mathcal{A}^2_{RM}, z) = \zeta_p(E_{CM}, z)$, where $\zeta_p(E_{CM}, z)$ is the local zeta function for the elliptic curve E_{CM}; the proof will be arranged into a series of lemmas 1-5.

Recall, that $\zeta_p(E_{CM}, z) = (1 - tr\ (\psi_{E(K)}(\mathfrak{P}))z + pz^2)^{-1}$, where $\psi_{E(K)}$ is the Grössencharacter on K, $\mathfrak{P}$ the prime ideal of K over p and tr is the trace of algebraic number [14], Ch.2, §9. Roughly, our proof consists in construction of representation ρ of $\psi_{E(K)}$ into the group of invertible elements (units) of $End\ (\tau(K_0(\mathcal{A}^2_{RM})))$, such that $tr\ (\psi_{E(K)}(\mathfrak{P})) = tr\ (\rho(\psi_{E(K)}(\mathfrak{P}))) = tr\ (A^p)$. This will be achieved with the help of an explicit formula for the Teichmüller functor F ([10], p.524):

$$F : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in End\ (E_{CM}) \longmapsto \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \in End\ (\mathbb{A}_{RM}). \tag{6}$$

Lemma 1 *Let $A = (a, b, c, d)$ be an integer matrix with $ad - bc \neq 0$ and $b = 1$. Then A is similar to the matrix $(a + d, 1, c - ad, 0)$.*

Proof. Indeed, consider a matrix $(1, 0, d, 1) \in SL_2(\mathbb{Z})$; it is verified directly, that the matrix realizes the required similarity. □

Lemma 2 *The matrix $A = (a+d, 1, c-ad, 0)$ is similar to its transpose $A^t = (a+d, c-ad, 1, 0)$.*

Proof. We shall use the following criterion: the (integer) matrices A and B are similar, if and only if the characteristic matrices $xI - A$ and $xI - B$ have the same Smith normal form. The calculation for the matrix $xI - A$ gives:

$$\begin{pmatrix} x-a-d & -1 \\ ad-c & x \end{pmatrix} \sim \begin{pmatrix} x-a-d & -1 \\ x^2-(a+d)x+ad-c & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & x^2-(a+d)x+ad-c \end{pmatrix},$$

where $\sim$ are the elementary operations between the rows (columns) of the matrix. Similarly, a calculation for the matrix $xI - A^t$ gives:

$$\begin{pmatrix} x-a-d & ad-c \\ -1 & x \end{pmatrix} \sim \begin{pmatrix} x-a-d & x^2-(a+d)x+ad-c \\ -1 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & x^2-(a+d)x+ad-c \end{pmatrix}.$$

Thus, $(xI - A) \sim (xI - A^t)$ and lemma 2 follows. $\square$

Corollary 1 *The matrices $(a, 1, c, d)$ and $(a+d, c-ad, 1, 0)$ are similar.*

Let E_{CM} be elliptic curve with complex multiplication by an order R in the ring of integers of the imaginary quadratic field k. Then $\mathbb{A}_{RM} = F(E_{CM})$ is a noncommutative torus with real multiplication by the order $\mathfrak{R}$ in the ring of integers of a real quadratic field $\mathfrak{k}$ (section 3). Let $tr\ (\alpha) = \alpha + \bar{\alpha}$ be the trace function of a (quadratic) algebraic number field.

Lemma 3 *Each $\alpha \in R$ goes under F into an $\omega \in \mathfrak{R}$, such that $tr\ (\alpha) = tr\ (\omega)$.*

Proof. Recall that each $\alpha \in R$ can be written in a matrix form for a given base $\{\omega_1, \omega_2\}$ of the lattice Λ. Namely, $\alpha\omega_1 = a\omega_1 + b\omega_2$ and $\alpha\omega_2 = c\omega_1 + d\omega_2$, where (a, b, c, d) is an integer matrix with $ad - bc \neq 0$. Note that $tr\ (\alpha) = a+d$ and $b\tau^2 + (a-d)\tau - c = 0$, where $\tau = \omega_2/\omega_1$. Since τ is an algebraic integer, we conclude that $b = 1$. In view of corollary 1, in a base $\{\omega_1', \omega_2'\}$, the α has a matrix form $(a+d, c-ad, 1, 0)$. To calculate $\omega \in \mathfrak{R}$ corresponding to α, we apply formula (6), which gives us:

$$F : \begin{pmatrix} a+d & c-ad \\ 1 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a+d & c-ad \\ -1 & 0 \end{pmatrix}. \tag{7}$$

In a given base $\{\lambda_1, \lambda_2\}$ of the pseudo-lattice $\mathbb{Z} + \mathbb{Z}\theta$ one can write $\omega\lambda_1 = (a+d)\lambda_1 + (c-ad)\lambda_2$ and $\omega\lambda_2 = -\lambda_1$. It is an easy exercise to verify that ω is a real quadratic integer with $tr\ (\omega) = a + d$; the latter coincides with the $tr\ (\alpha)$. $\square$

Let $\omega \in \mathfrak{R}$ be an endomorphism of the pseudo-lattice $\mathbb{Z} + \mathbb{Z}\theta$ of degree $deg\ (\omega) := \omega\bar{\omega} = n$. The endomorphism maps pseudo-lattice to a sub-lattice of index n. Any such has a form $\mathbb{Z} + (n\theta)\mathbb{Z}$ [3], p.131. Let us calculate ω in a base $\{1, n\theta\}$, when ω is given by the matrix $(a+d, c-ad, -1, 0)$. In this case $n = c - ad$ and ω induces an automorphism $\omega^* = (a+d, 1, -1, 0)$ of the sublattice $\mathbb{Z} + (n\theta)\mathbb{Z}$ according to the matrix equation:

$$\begin{pmatrix} a+d & n \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} a+d & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ n\theta \end{pmatrix}. \tag{8}$$

Thus, one gets a map $\rho : \mathfrak{R} \to \mathfrak{R}^*$ given by the formula $\omega = (a+d, n, -1, 0) \mapsto \omega^* = (a+d, 1, -1, 0)$, where $\mathfrak{R}^*$ is the group of units of $\mathfrak{R}$. Since $tr\ (\omega^*) = a + d = tr\ (\omega)$ and $\omega^* = \rho(\omega)$, one gets the following

Corollary 2 *For all $\omega \in \mathfrak{R}$, it holds $tr\ (\omega) = tr\ (\rho(\omega))$.*

Note, that $\mathfrak{R}^* = \{\pm\varepsilon^k \mid k \in \mathbb{Z}\}$, where $\varepsilon > 1$ is a fundamental unit of the order $\mathfrak{R} \subseteq O_{\mathfrak{k}}$; here $O_{\mathfrak{k}}$ means the ring of integers of a real quadratic field $\mathfrak{k} = \mathbb{Q}(\theta)$. Choosing a sign in front of ε^k, the following index map is defined $\iota : R \xrightarrow{\text{F}} \mathfrak{R} \xrightarrow{\rho} \mathfrak{R}^* \longrightarrow \mathbb{Z}$. Let $\alpha \in R$ and $deg\ (\alpha) = -n$. To calculate the $\iota(\alpha)$, recall some notation from Hasse [6], §16.5.C. Let $\mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. For brevity, let $I = \mathbb{Z} + \mathbb{Z}\theta$ be a pseudo-lattice and $I_n = \mathbb{Z} + (n\theta)\mathbb{Z}$ its sub-lattice of index n; the fundamental units of I and I_n are ε and ε_n, respectively. By $\mathfrak{G}_n$ one understands a subgroup of $\mathbb{Z}/n\mathbb{Z}$ of prime residue classes $mod\ n$. The $\mathfrak{g}_n \subset \mathfrak{G}_n$ is a subgroup of the non-zero divisors of the $\mathfrak{G}_n$. Finally, let g_n be the smallest number, such that it divides $|\mathfrak{G}_n/\mathfrak{g}_n|$ and $\varepsilon^{g_n} \in I_n$. (The notation drastically simplifies in the case $n = p$ is a prime number.)

Lemma 4 *$\iota(\alpha) = g_n$.*

Proof. Notice, that $deg\ (\omega) = -deg\ (\alpha) = n$, where $\omega = F(\alpha)$. Then the map ρ defines I and I_n; one can now apply the calculation of [6], pp 296-300. Namely, Theorem $XIII'$ on p. 298 yields the required result. (We kept the notation of the original.) $\square$

Corollary 3 *$\iota(\psi_{E(K)}(\mathfrak{P})) = p$.*

Proof. It is known, that $deg\ (\psi_{E(K)}(\mathfrak{P})) = -p$, where $\psi_{E(K)}(\mathfrak{P}) \in R$ is the Grössencharacter. To calculate the g_n in the case $n = p$, notice that the $\mathfrak{G}_p \cong \mathbb{Z}/p\mathbb{Z}$ and $\mathfrak{g}_p$ is trivial. Thus, $|\mathfrak{G}_p/\mathfrak{g}_p| = p$ is divisible only by 1 or p. Since ε^1 is not in I_n, one concludes that $g_p = p$. The corollary follows. $\square$

Lemma 5 *$tr\ (\psi_{E(K)}(\mathfrak{P})) = tr\ (A^p)$.*

Proof. It is not hard to see, that A is a hyperbolic matrix with the eigenvector $(1, \theta)$; the corresponding (Perron-Frobenius) eigenvalue is a fundamental unit $\varepsilon > 1$ of the pseudo-lattice $\mathbb{Z} + \mathbb{Z}\theta$. In other words, A is a matrix form of the algebraic number ε. It is immediate, that A^p is the matrix form for the ε^p and $tr\ (A^p) = tr\ (\varepsilon^p)$. In view of lemma 3 and corollary 3, $tr\ (\alpha) = tr\ (F(\alpha)) = tr\ (\rho(F(\alpha)))$ for $\forall\alpha \in R$. In particular, if $\alpha = \psi_{E(K)}(\mathfrak{P})$ then, by corollary 3, one gets $\rho(F(\psi_{E(K)}(\mathfrak{P}))) = \varepsilon^p$. Taking traces in the last equation, we obtain the conclusion of lemma 5. $\square$

The fact $\zeta_p(\mathcal{A}^2_{RM}, z) = \zeta_p(E_{CM}, z)$ follows from lemma 5, since the trace of the Grössencharacter coincides with such for the matrix A^p. Let E_{CM} be an elliptic curve with complex multiplication by an order in the imaginary quadratic field k and K the minimal field of definition of the E_{CM}.

Lemma 6 *$L(E_{CM}, s) \equiv L(\sigma_2, s)$, where $L(E_{CM}, s)$ is the Hasse-Weil L-function of E_{CM} and $L(\sigma_2, s)$ the Artin L-function for an irreducible representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$.*

Proof. By the Deuring theorem (see e.g. [14], p.175), $L(E_{CM}, s) = L(\psi_K, s)L(\overline{\psi}_K, s)$, where $L(\psi_K, s)$ is the Hecke L-series attached to the Grössencharacter $\psi : \mathbb{A}^*_K \to \mathbb{C}^*$; here $\mathbb{A}^*_K$ denotes the adele ring of the field K and the bar means a complex conjugation. Notice, that since our elliptic curve has complex multiplication, the group $Gal\ (K|k)$ is abelian; one can apply Theorem 5.1 [7], which says that the Hecke L-series $L(\sigma_1 \circ \theta_{K|k}, s)$ equals the Artin L-function $L(\sigma_1, s)$, where $\psi_K = \sigma \circ \theta_{K|k}$ is the Grössencharacter and $\theta_{K|k} : \mathbb{A}^*_K \to Gal\ (K|k)$ the canonical homomorphism. Thus, one gets $L(E_{CM}, s) \equiv L(\sigma_1, s)L(\overline{\sigma}_1, s)$, where $\overline{\sigma}_1 : Gal\ (K|k) \to \mathbb{C}$ means a (complex) conjugate representation of the Galois group. Consider the local factors of the Artin L-functions $L(\sigma_1, s)$ and $L(\overline{\sigma}_1, s)$; it is immediate, that they are $(1 - \sigma_1(Fr_p)p^{-s})^{-1}$ and $(1 - \overline{\sigma}_1(Fr_p)p^{-s})^{-1}$, respectively.

Let us consider a representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$, such that

$$\sigma_2(Fr_p) = \begin{pmatrix} \sigma_1(Fr_p) & 0 \\ 0 & \overline{\sigma}_1(Fr_p) \end{pmatrix}. \tag{9}$$

It can be verified, that $det^{-1}(I_2 - \sigma_2(Fr_p)p^{-s}) = (1 - \sigma_1(Fr_p)p^{-s})^{-1}(1 - \overline{\sigma}_1(Fr_p)p^{-s})^{-1}$, i.e. $L(\sigma_2, s) = L(\sigma_1, s)L(\overline{\sigma}_1, s)$. Lemma 6 follows. $\square$

By lemma 6, we conclude, that $L(\mathcal{A}^2_{RM}, s) \equiv L(\sigma_2, s)$ for an irreducible representation $\sigma_2 : Gal\ (K|k) \to GL_2(\mathbb{C})$. It remains to notice that $L(\sigma_2, s) = L(\sigma_2', s)$, where $\sigma_2' : Gal\ (K|\mathbb{Q}) \to GL_2(\mathbb{C})$ [1], §3. Case $n = 1$ of theorem 1 follows. $\square$

2.2 Case $n = 0$

When $n = 0$, one gets a one-dimensional (degenerate) noncommutative torus; such an object, $\mathcal{A}_{\mathbb{Q}}$, can be obtained from the 2-dimensional torus $\mathcal{A}^2_\theta$ by forcing $\theta = p/q \in \mathbb{Q}$ be a rational number (hence our notation). One can always assume $\theta = 0$ and, thus, $\tau(K_0(\mathcal{A}_{\mathbb{Q}})) = \mathbb{Z}$. To calculate matrix L^1_p, notice that the group of automorphisms of the $\mathbb{Z}$-module $\tau(K_0(\mathcal{A}_{\mathbb{Q}})) = \mathbb{Z}$ is trivial, i.e. is a multiplication by ± 1; hence our 1×1 (real) matrix A is either 1 or -1. Since A must be positive, we conclude, that $A = 1$. However, $A = 1$ is not a prime matrix, if one allows the complex entries; indeed, for any $N > 1$ matrix $A' = \zeta_N$ gives us $A = (A')^N$, where $\zeta_N = e^{\frac{2\pi i}{N}}$ is the N-th root of unity. Therefore, $A = \zeta_N$ and $L^1_p = tr\ (A^p) = A^p = \zeta^p_N$. A degenerate noncommutative torus, corresponding to the matrix $A = \zeta_N$, we shall write as $\mathcal{A}^N_{\mathbb{Q}}$; in turn, such a torus is the image (under the Teichmüller functor) of a zero-dimensional abelian variety, which we denote by V^N_0. Suppose that $Gal\ (K|\mathbb{Q})$ is abelian and let $\sigma : Gal\ (K|\mathbb{Q}) \to \mathbb{C}^\times$ be a homomorphism. Then, by the Artin reciprocity [5], there exists an integer N_σ and a Dirichlet character $\chi_\sigma : (\mathbb{Z}/N_\sigma\mathbb{Z})^\times \to \mathbb{C}^\times$, such that $\sigma(Fr_p) = \chi_\sigma(p)$; choose our zero-dimensional variety be $V^{N_\sigma}_0$. In view of the notation, $L^1_p = \zeta^p_{N_\sigma}$; on the other hand, it is verified directly, that $\zeta^p_{N_\sigma} = e^{\frac{2\pi i}{N_\sigma}p} = \chi_\sigma(p)$. Thus, $L^1_p = \chi_\sigma(p)$. To obtain a local zeta function, we substitute $a_1 = L^1_p$ into the formula (4) and get

$$\zeta_p(\mathcal{A}^{N_\sigma}_{\mathbb{Q}}, z) = \frac{1}{1 - \chi_\sigma(p)z}, \tag{10}$$

where $\chi_\sigma(p)$ is the Dirichlet character. Therefore, $L(\mathcal{A}^{N_\sigma}_{\mathbb{Q}}, s) \equiv L(s, \chi_\sigma)$ is the Dirichlet L-series; such a series, by construction, coincides with the Artin

L-series of the representation $\sigma : Gal\ (K|\mathbb{Q}) \to \mathbb{C}^{\times}$. Case $n = 0$ of theorem 1 follows. $\square$

3 Teichmüller functors

Denote by Λ a lattice of rank $2n$; recall, that an n-dimensional (principally polarized) abelian variety, V_n, is the complex torus $\mathbb{C}^n/\Lambda$, which admits an embedding into a projective space [9].

3.1 Abelian varieties of dimension $n = 1$

A. Basic example. Let $n = 1$ and consider the complex torus $V_1 \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$; it always embeds (via the Weierstrass $\wp$ function) into a projective space as a non-singular elliptic curve. Let $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and $\partial\mathbb{H} = \{\theta \in \mathbb{R} \mid y = 0\}$ its (topological) boundary. We identify $V_1(\tau)$ with the points of $\mathbb{H}$ and $\mathcal{A}_\theta^2$ with the points of $\partial\mathbb{H}$. Let us show, that the boundary is natural; the latter means, that the action of the modular group $SL_2(\mathbb{Z})$ extends to the boundary, where it coincides with the stable isomorphisms of tori. Indeed, conditions (1) are equivalent to

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \quad (11)$$

where $ad-bc = 1$, $a, b, c, d \in \mathbb{Z}$ and $\Theta' = (A\Theta+B)/(C\Theta+D) = (0, \frac{a\theta+b}{c\theta+d}, -\frac{a\theta+b}{c\theta+d}, 0)$. Therefore, $\theta' = (a\theta + b)(c\theta + d)^{-1}$ for a matrix $(a, b, c, d) \in SL_2(\mathbb{Z})$. Thus, the action of $SL_2(\mathbb{Z})$ extends to the boundary $\partial\mathbb{H}$, where it induces stable isomorphisms of the noncommutative tori.

B. The Teichmüller functor ([10]). There exists a continuous map $F_1 : \mathbb{H} \to \partial\mathbb{H}$, which sends isomorphic complex tori to the stably isomorphic noncommutative tori. An exact result is this. Let ϕ be a closed form on the torus, whose trajectories define a measured foliation; according to the Hubbard-Masur theorem (applied to the complex tori), this foliation corresponds to a point $\tau \in \mathbb{H}$. The map $F_1 : \mathbb{H} \to \partial\mathbb{H}$ is defined by the formula $\tau \mapsto \theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi$, where γ_1 and γ_2 are generators of the first homology of the torus. The following is true: (i) $\mathbb{H} = \partial\mathbb{H} \times (0, \infty)$ is a trivial fiber bundle, whose projection map coincides with F_1; (ii) F_1 is a functor, which sends isomorphic complex tori to the stably isomorphic noncommutative tori. We

shall refer to F_1 as the *Teichmüller functor*. Recall, that the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is said to have a complex multiplication, if the endomorphism ring of the lattice $\Lambda=\mathbb{Z}+\tau\mathbb{Z}$ exceeds the trivial ring $\mathbb{Z}$; the complex multiplication happens if and only if τ is an algebraic number in an imaginary quadratic field. The following is true: $F_1(V_1^{CM})=\mathcal{A}_{RM}^2$, where V_1^{CM} is a torus with complex multiplication.

3.2 Abelian varieties of dimension $n\ge 1$

A. The Siegel upper half-space ([9]). The space $\mathbb{H}_n := \{\tau=(\tau_i)\in \mathbb{C}^{\frac{n(n+1)}{2}} \mid Im\ (\tau_i)>0\}$ of symmetric $n\times n$ matrices with complex entries is called a *Siegel upper half-space*; the points of $\mathbb{H}_n$ are one-to-one with the n-dimensional principally polarized abelian varieties. Let $Sp(2n,\mathbb{R})$ be the symplectic group; it acts on $\mathbb{H}_n$ by the linear fractional transformations $\tau\to\tau'=(a\tau+b)/(c\tau+d)$, where $(a,b,c,d)\in Sp(2n,\mathbb{R})$ and a,b,c and d are the $n\times n$ matrices with real entries. The abelian varieties V_n and V_n' are isomorphic, if and only if, τ and τ' belong to the same orbit of the group $Sp(2n,\mathbb{Z})$; the action is discontinuous on $\mathbb{H}_n$ [9], Ch.2, §4. Denote by Σ_{2n} a space of the $2n$-dimensional normal noncommutative tori. The following lemma is critical.

Lemma 7 *$Sp(2n,\mathbb{R})\subseteq O(n,n|\mathbb{R})$.*

Proof. (i) The group $O(n,n\mid\mathbb{R})$ can be defined as a subgroup of $GL_2(\mathbb{R})$, which preserves the quadratic form $f(x_1,\dots,x_{2n})=x_1x_{n+1}+x_2x_{n+2}+\dots+x_nx_{2n}$ [13]. We shall denote $u_i=x_1$ for $1\le i\le n$ and $v_i=x_i$ for $n+1\le i\le 2n$. Consider the following skew symmetric bilinear form $q(u,v)=u_1v_{n+1}+\dots+u_nv_{2n}-u_{n+1}v_1-\dots-u_{2n}v_n$, where $u,v\in\mathbb{R}^{2n}$. It is known, that each linear substitution $g\in Sp(2n,\mathbb{R})$ preserves the form $q(u,v)$. Since $q(u,v)=f(x_1,\dots,x_{2n})-u_{n+1}v_1-\dots-u_{2n}v_n$, one concludes that g also preserves the form $f(x_1,\dots,x_{2n})$, i.e. $g\in O(n,n|\mathbb{R})$. It is easy to see, that the inclusion is proper except the case $n=1$, i.e. when $Sp(2,\mathbb{R})\cong O(1,1|\mathbb{R})\cong SL_2(\mathbb{R})$. Lemma follows.

(ii) We wish to give a second proof of this important fact, which is based on the explicit formulas for the block matrices A,B,C and D. The fact that a symplectic linear transformation preserves the skew symmetric bilinear form

$q(u,v)$ can be written in a matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{12}$$

where t is the transpose of a matrix. Performing the matrix multiplication, one gets the following matrix identities $a^t d - c^t b = I,\ a^t c - c^t a = 0 = b^t d - d^t b$. Let us show, that these identities imply the Rieffel-Schwarz identities (1) imposed on the matrices A, B, C and D. Indeed, in view of the formulas (11), the Rieffel-Schwarz identities can be written as:

$$\left\{ \begin{array}{llll} \begin{pmatrix} a^t & 0 \\ 0 & a^t \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} & + \begin{pmatrix} 0 & c^t \\ -c^t & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} & = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} a^t & 0 \\ 0 & a^t \end{pmatrix} \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} & + \begin{pmatrix} 0 & c^t \\ -c^t & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} & = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -b^t \\ b^t & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} & + \begin{pmatrix} d^t & 0 \\ 0 & d^t \end{pmatrix} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} & = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{array} \right. \tag{13}$$

A step by step matrix multiplication in (13) shows that the identities $a^t d - c^t b = I,\ a^t c - c^t a = 0 = b^t d - d^t b$ imply the identities (13). (Beware: the operation is not commutative.) Thus, any symplectic transformation satisfies the Rieffel-Schwarz identities, i.e. belongs to the group $O(n, n|\mathbb{R})$. Lemma 7 follows. $\square$

B. The generalized Teichmüller functors. By lemma 7, the action of $Sp(2n, \mathbb{Z})$ on the $\mathbb{H}_n$ extends to the Σ_{2n}, where it acts by stable isomorphisms of the noncommutative tori; thus, Σ_{2n} is a natural boundary of the Siegel upper half-space $\mathbb{H}_n$. However, unless $n = 1$, the Σ_{2n} is *not* a topological boundary of $\mathbb{H}_n$. Indeed, $dim_{\mathbb{R}}(\mathbb{H}_n) = n(n+1)$ and $dim_{\mathbb{R}}(\partial\mathbb{H}_n) = n^2 + n - 1$, while $dim_{\mathbb{R}}(\Sigma_{2n}) = n$. Thus, Σ_{2n} is an n-dimensional subspace of the topological boundary of $\mathbb{H}_n$; this subspace is everywhere dense in $\partial\mathbb{H}_n$, since the $Sp(2n, \mathbb{Z})$-orbit of an element of Σ_{2n} is everywhere dense in $\partial\mathbb{H}_n$ [13]. A (conjectural) continuous map $F_n : \mathbb{H}_n \to \Sigma_{2n}$, we shall call a *generalized Teichmüller functor*. The F_n has the following properties: (i) it sends each pair of isomorphic abelian varieties to a pair of the stably isomorphic even dimensional normal tori; (ii) the range of F_n on the abelian varieties with complex multiplication consists of the noncommutative tori with real multiplication. As explained, such a functor has been constructed only in the case $n = 1$; the difficulties in higher dimensions are due to the lack (so far) of a proper Teichmüller theory for the abelian varieties of dimension $n \geq 2$.

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